Exam C (Part I) **Name**

Please Note: If you wish full credit for your work, provide complete and efficient explanations. Write clearly, and organize your work well. Suggestion: Do your initial thinking on scrap paper. Only thereafter proceed to the test making careful use of the space designated for the problem. Calculators can be used only in elementary arithmetic and trig mode.

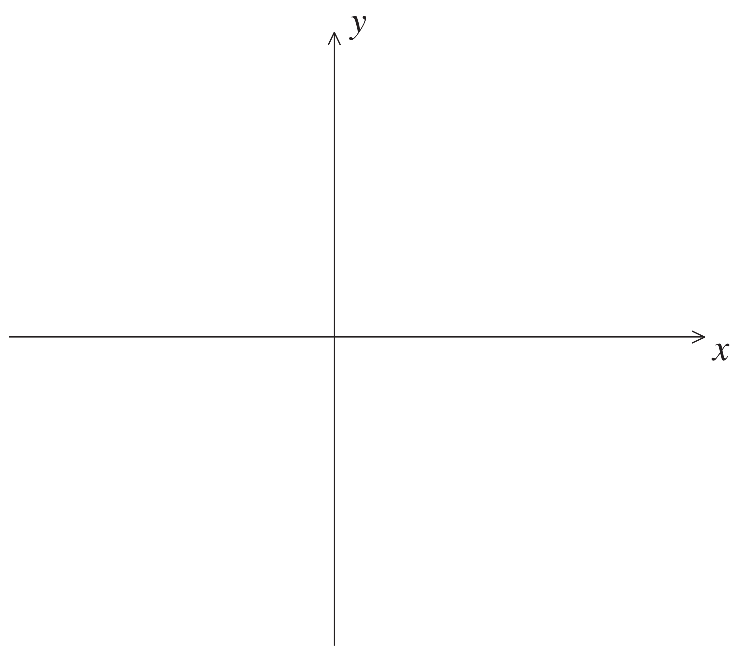
1. Let r_M and r_S be the radii of the Moon and the Sun respectively, and let D_M and D_S be the distances from the Earth to the Moon and Sun respectively. A Greek philosopher looks out at the sky and sees a solar eclipse (the precise time on which the Moon just barely but completely blocks out the light coming from the Sun). He is aware of the estimate of $2°$ for the angular diameters of both the Sun and the Moon. He draws a very careful diagram of what he observes and correctly writes down all the information about r_M, r_S, D_M, and D_S that his diagram provides. What diagram did he draw and what information did he write down?

2. Draw a circle as well as a diameter AB carefully. Extend the diameter to a triangle ΔABC in such a way that the vertex C is on the circle. Show that the angle at C is $90°$.

3. Consider the ellipse $\frac{x^2}{3^2} + y^2 = 1$ and the parabola $y = \frac{5}{9}x^2 - 5$.

3a. What are the semimajor and semiminor axes of the ellipse and what is the lowest point on the parabola.

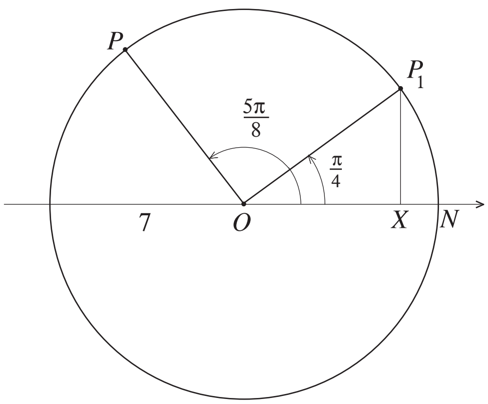
3b. Determine all the points of intersection of the graphs of the two equations. (Give some thought to the most efficient substitution). Sketch both of the graphs including all the points of intersection on the coordinate plane below.

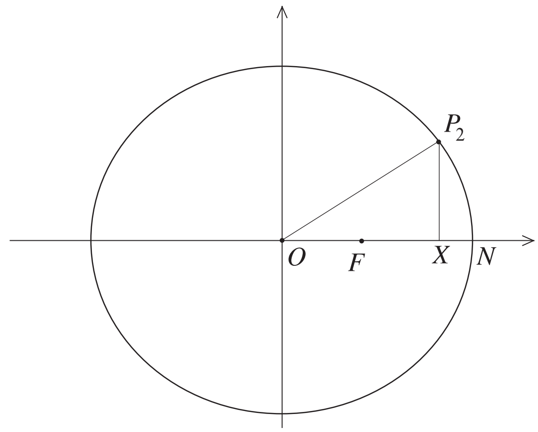


3c. Sketch the region $R = \{(x, y) \mid \frac{5}{9}x^2 - 5 \leq y \leq \frac{1}{3}\sqrt{3^2 - x^2}\}$ into your diagram.

3d. Determine the area of the region R.

4. The graphs of the circle of radius 7 with center at O and that of the ellipse $\frac{x^2}{7^2} + \frac{y^2}{6^2} = 1$ are sketched below. The two points P and P_1 on the circle are as indicated. The two points X on the two x-axes are the same point. The segments P_1X and P_2X are both perpendicular to the x-axis.





The point F is one of the focal points of the ellipse.

4a. Determine the coordinates of the point P on the circle.

4b. Use the fact that the point P_1 on the circle is the point $(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2})$, (i) to determine the area of the circular slice NXP_1 and (ii) to determine the coordinates of the point P_2 on the ellipse.

4c. Determine the area of the elliptical slice NXP_2 and then determine the area of the sector NFP_2.

5. Consider the function $f(x) = 9 - x^2$ with $0 \leq x \leq 2$. select the points $0 \leq 0.5 \leq 1 \leq 1.5 \leq 2$ on the x-axis between 0 and 2.

5a. Compute an approximation of the area under the graph of $f(x)$ over $0 \leq x \leq 2$ by using the approach of Leibniz in Section 5.6.

5b. Use Archimedes's theorem to find the area precisely.

5c. Use the fundamental theorem of calculus to find the area.

6. A particle of mass 1 unit is moving along a coordinate axis. Its position on the axis at any time $t \geq 0$ is given by $p(t) = t^3 - 12t^2 + 30t$. Consider any instant $t \geq 0$ during the motion of the particle.

i. Compute the average velocity over the time interval $[0, t]$. Compute the velocity of the particle at the instant t.

ii. Compute the average acceleration over the time interval $[0, t]$. Compute the acceleration of the particle at the instant t. At what times is the force that propels the particle equal to zero?

7. A point moves in the xy-plane. It starts at $(-4, 2)$ at time $t = 0$. At any time $t \geq 0$ its position is given by $(x(t), y(t))$. It is known that at any time t its velocities in the x and y directions are $5t$ and $3t^2 + 4t$ respectively.
7a. Determine the function $x(t)$ and $y(t)$ in precise terms. Where is the point at time $t = 2$?

7b. Determine an equation for the path of the point (i.e. express y as a function of x), sketch a graph of such function, explain how the point moves and discuss the shape of the path.

7c. Suppose that the point has a mass of 1 unit, what are the forces acting on it?

8. A particle P of mass m is in orbit around a fixed point S. The only force acting on P is a centripetal force in the direction of S. Let κ be Kepler's constant for the orbit. Let the particle be in a typical position P. Suppose that the particle moves from position P to position Q and that a small time Δt elapses during this motion. Take the point T on the segment SP obtained by dropping a perpendicular from Q. And let R be the intersection of the tangent to the orbit at P and the line that goes through Q and is parallel to SP.

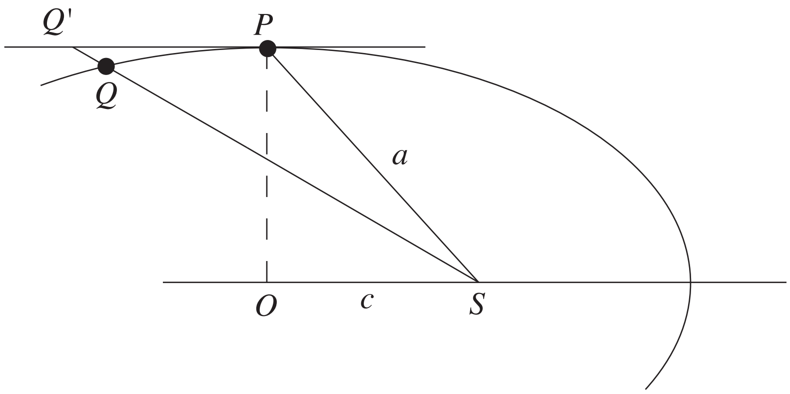
i. Draw a careful diagram of the situation just described.

ii. Let r_P be the distance from P to S and let F_P be the magnitude of the centripetal force on the particle. Verify, providing clear explanations for your steps, that $F_P = \frac{8\kappa^2 m}{r_P^2} \lim_{Q \to P} \frac{QR}{QT^2}$. You may use the fact that a constant force F acting for a time Δt on a particle of mass m that is initially at rest, will move the particle a distance of $\frac{F}{2m} \cdot (\Delta t)^2$.

8. A particle of constant mass m moves on the x-axis. It starts at time $t = 0$ from the origin with zero initial velocity and is driven by a constant force of magnitude F in the direction of the positive x-axis. Its position on the x-axis at any time $t \geq 0$ is $x(t)$. Express the function $x(t)$ in terms of the constants mentioned above and t.

$x(t) =$

11. (15 pts) The figure below considers a planet in an elliptical orbit around the Sun S. The point O is the center of the ellipse. The semimajor and semiminor axes of the ellipse are a and b respectively, and T is the period of the orbit. Consider the planet at position P at the "top" of its orbit and again at a small time Δt later at Q. The point Q' is the point of intersection of the tangent to the orbit at P and the extension of the segment SQ. Let $\Delta s = Q'P$ and make use of Kepler's constant $\kappa = \frac{ab\pi}{T}$ to approximate the average velocity $\frac{\Delta s}{\Delta t}$ of the planet from P to Q. Then show that the velocity of the planet at P is equal to $v_P = \frac{2\pi a}{T}$.



Formulas and Facts:

Area of circular sector equals: $\frac{1}{2}\theta r^2$
Archimedes's theorem: Area of parabolic section $= \frac{4}{3} \times$ Area of inscribed triangle.

Cavalieri's principle: Consider two regions in the plane and let C and D be their areas. Let c_x and d_x be the respective cross-sections of the two regions for all x on some coordinate axis with $a \leq x \leq b$. If $c_x = kd_x$ for all x and some constant k, then $C = kD$.

$F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2}$ $\quad F = G\frac{Mm}{r^2}$ $\quad \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$